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The Effects of Field Errors on Low-Gain Free Electron Lasers

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The effects of random wiggler magnetic field errors on low-gain free electron lasers are examined analytically and numerically through the use of ensemble averaging techniques. Wiggler field errors perturb the electron beam as it propagates and lead to a random walk of the beam centroid δx, variations in the axial beam energy $\delta\gamma_z$ and deviations in the relative phase of the electrons in the ponderomotive wave $\delta\psi$. In principle, the random walk may be kept as small as desired through the use of transverse focusing and beam steering. Transverse focusing of the electron beam is shown to be ineffective in reducing the phase deviation. Furthermore, it is shown that beam steering at the wiggler entrance reduces the average phase deviation at the end of the wiggler by 1/3. The effect of the field errors (via the phase deviation) on the gain in the low-gain regime is calculated. To avoid significant reduction in gain it is ncessary for the phase deviation to be small compared to 2π. The detrimental effects of wiggler errors on low-gain free electron lasers may be reduced by arranging the magnet poles in an optimal ordering such that the magnitude of the phase deviation is minimized.

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CONTENTS

THE EFFECTS OF FIELD ERRORS ON LOW-GAIN FREE ELECTRON LASERS

I. Introduction

Intrinsic magnetic field errors δB are present in any realistic wiggler magnet. Such errors are unavoidable and arise from imperfections in the fabrication and assembly of wiggler magnets. State–of–the–art wiggler construction techniques yield rms field errors on the order[1] $(\delta B/B_w)_{rms} \simeq 0.1 - 0.5\%$. These field errors perturb the electron beam as it propagates through the wiggler[2–16] and lead to i) a random walk of the beam centroid, δx, ii) variations in the axial beam energy, $\delta\gamma_z$, and iii) variations in the relative phase of the electrons in the ponderomotive potential, $\delta\psi$. If left uncorrected, field errors ultimately decrease free electron laser (FEL) gain[2–16] (this reduction becomes more significant for long wigglers). Reduction in gain may occur from a loss of overlap between the radiation and electron beam (due to large δx) or from a loss of FEL resonance (due to large $\delta\psi$).

The initial research on the effects of field errors, for the most part, was primarily concerned with the random walk δx. It has been shown that the random walk δx may be effectively controlled by i) transverse beam focusing[3–9] (finite k_β, where k_β is the betatron wavenumber) and by ii) periodic beam steering.[2–10] By using either one or a combination of beam focusing and periodic steering, in principle, the random walk δx may be kept as small as desired. The major conclusions of the present work are the following. Given that the random walk δx may be effectively controlled, the phase deviation $\delta\psi$ is the primary physical parameter characterizing loss of gain for FELs in the low–gain regime.[3–16] In particular, in order to avoid significant reduction in gain, it is necessary that $|\delta\psi| << 2\pi$. In addition, transverse beam focusing is not effective in controlling $\delta\psi$. Specifically, it may be shown that the mean phase deviation $\langle\delta\psi\rangle$ is independent of transverse focusing (independent of k_β), where $\langle ... \rangle$ signifies an ensemble average. Furthermore, beam steering[2–10] may be used to reduce $|\delta\psi|$ when[8] $L_S < \lambda_\beta$, where L_S is the length over which the steering is performed and $\lambda_\beta = 2\pi/k_\beta$. As an example, for $k_\beta = 0$ and one steering segment, $\langle\delta\psi\rangle = (1/3)\langle\delta\psi\rangle_N$, where $\langle\delta\psi\rangle_N$ is the value in the absence of steering.

As a further motivation, it is appropriate to consider some aspects of wiggler design. Typically, when "ordering" a wiggler from a vendor, limits are placed on $\delta B_{rms} \equiv \langle\delta B^2\rangle^{1/2}$ and $|\int dz\delta B|$. To meet these specifications, the vendor may "arrange" the magnet poles

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(i.e., actual pole rearrangements, the use of shims, judicious magnet selection, etc.) in an optimum sequence[12-15] such that $|\int dz\delta B|$ is minimized. This and other research[3-16] indicates, however, that for low-gain FELs the optimum "figure of merit" to minimize is not the line integral $|\int dz\delta B|$, but the magnitude of the phase deviation $|\delta\psi|$.

In the following, the effects of random transverse magnetic field errors, $\delta B_{\perp}(z)$, on the performance of low-gain FELs are studied analytically and numerically. In particular, the transverse displacement, parallel energy variation and relative phase deviation of an electron beam propagating through a wiggler are calculated neglecting the effects of finite beam emittance, initial beam energy spread and wiggler field tapering. Furthermore, the FEL gain in the low-gain regime is determined in the 1D limit, assuming a plane wave, non-diffracting radiation field. Expressions are derived for a particular FEL quantity Q for a single wiggler realization (a specific set of field errors) and for an ensemble of statistically identical wigglers. The ensemble averages (the mean and the variance) of the quantity Q are useful for determining the most probable range of Q for a particular member of the ensemble. The remainder of this paper is organized as follows. The random walk of the beam centroid and the consequent variations in the axial beam energy are discussed in Sections II and III, respectively. The deviations in the relative phase resulting from the field errors are examined in Section IV. In Section V, the effect of the field errors on the FEL gain in the low-gain regime is determined. The benefits of beam steering are analyzed in Section VI and addition methods for reducing the detrimental effects of field errors are discussed in Section VII. This paper concludes with a discussion and summary in Section VIII.

II. Transverse Beam Centroid Deviations

As the electron beam propagates through the wiggler, the electrons experience random velocity kicks $\delta v_\perp$ via the $v_z \times \delta B_\perp$ random force. The transverse centroid motion of an electron beam passing through a wiggler with transverse gradients (weak focusing) and finite field errors is characterized by an equation of the form[6]

$$d^2\delta x/dz^2 = -k_\beta^2 \delta x + k_w a_w \delta \hat{B}_y/\gamma, \tag{1}$$

where k_w is the wiggler wavenumber, $k_\beta = k_w a_w/(\sqrt{2}\gamma)$ is the betatron wavenumber, $\delta\hat{B}_y = \delta B_y/B_w$ is the normalized field error, B_w is the ideal wiggler peak magnetic field, $a_w = eB_w/k_w mc^2$, γ is the relativistic factor of the electron beam and z is the axial propagation distance. The first term on the right represents the focusing force due to the transverse gradients in the wiggler field, whereas the second term on the right represents the random force due to the field errors. This equation may be solved to give the random centroid motion[6]

$$\delta\beta_x = \frac{a_w k_w}{\gamma}\int_0^z dz' \cos k_\beta(z'-z)\delta\hat{B}_y(z'), \tag{2}$$

$$\delta x = -\frac{a_w k_w}{\gamma k_\beta}\int_0^z dz' \sin k_\beta(z'-z)\delta\hat{B}_y(z'), \tag{3}$$

where $\delta\beta_x = \delta v_x/c$ is the normalized transverse velocity deviation.

Given the precise functional dependence of the wiggler errors $\delta B_y(z)$ for a given wiggler, the above expressions may be used to calculate the transverse orbit deviations $\delta\beta_x(z)$ and $\delta x(z)$ for that specific wiggler. However, one does not always know ahead of time the full functional dependence of $\delta B_y(z)$. Instead, one may know only certain statistical properties of the field errors, such as the rms value δB_{rms}. Hence, it is useful to consider an ensemble of statistically identical wigglers for which the statistical properties of the field errors are known. By performing appropriate averages over this ensemble, one may determine the mean $\langle Q\rangle$ and variance σ for a quantity Q and, hence, determine the most probable range of a single realization of Q. Here and throughout the following, $\delta B_y(z)$ is assumed[6] to be a random, homogeneous function with zero mean, finite variance and with an autocorrelation distance given by z_{cy} ($z_{cy} \simeq \lambda_w/2$ is assumed). Details of the statistical properties of of the field errors $\delta B_y(z)$ are discussed in Ref. 6.

Statistically averaging over an ensemble of wigglers, it is possible to determine the mean-square centroid motion[6] (neglecting the effects of finite beam emittance)

$$\langle \delta\beta_x^2 \rangle = D\left(z + \frac{\sin 2k_\beta z}{2k_\beta}\right), \tag{4}$$

$$\langle \delta x^2 \rangle = \frac{D}{k_\beta^2}\left(z - \frac{\sin 2k_\beta z}{2k_\beta}\right), \tag{5}$$

where $D = a_w^2 k_w^2 \langle \delta \hat{B}_y^2 \rangle z_{cy}/(2\gamma^2)$. Physically, the centroid orbits δx and $\delta\beta_x$ represent diffusing betatron orbits characterized by a diffusion coefficient D. Notice that by increasing k_β^2 by additional external focusing, one may, in principle, keep δx_{rms} as small as desired. (The minimum centroid displacement is limited by finite emittance effects, which are neglected in the present discussion.) Furthermore, notice that in the 1D limit, $(2k_\beta z)^2 << 1$, $\langle \delta\beta_x^2 \rangle = 2Dz$ and $\langle \delta x^2 \rangle = 2Dz^3/3$, as found previously by Kincaid.[2] Hence, weak focusing (finite k_β) is effective in reducing the asymptotic scaling of the random walk δx_{rms} from $z^{3/2}$ to $z^{1/2}$. To avoid loss of the overlap between the radiation and electron beam, it is desirable to keep $\langle \delta x^2 \rangle << r_s^2$, where r_s is the radiation spot size. A detailed discussion of transverse orbit deviations arising from random field errors in various wiggler configurations is given in Ref. 6 and summarized in Appendix A.

III. Axial Beam Energy Variations

Not only do the field errors perturb the transverse motion of the electrons, they also perturb the axial motion. This is true since a static magnetic field conserves total electron energy. The axial motion may easily be calculated[6] using the above expressions for the transverse motion along with $\beta_z^2 + \beta_\perp^2$ =constant. One may calculate various statistical moments of the axial motion, such as the mean axial energy variation $\langle\delta\gamma_z\rangle = \langle\gamma_z\rangle - \gamma_{z0}$. For example, the mean energy variation for a helical wiggler with transverse focusing is given by

$$\frac{\langle\delta\gamma_z\rangle}{\gamma_{z0}} = -\frac{(1+a_w^2/4)}{2(1+a_w^2)^2} a_w^2 k_w^2 \left(\langle\delta\hat{B}_x^2\rangle + \langle\delta\hat{B}_y^2\rangle\right) z_c z, \tag{6}$$

where the limit $(2k_\beta z)^2 >> 1$ has been assumed and $z_c = z_{cx} = z_{cy}$. A detailed discussion of the axial energy variation for various wiggler configurations is given in Ref. 6.

Statistically, $\langle\delta\gamma_z\rangle$ may be interpreted as an effective energy spread due to field errors.[6] This effective energy spread may lead to a loss of FEL resonance. Heuristically, in order to maintain resonance, one expects that in the low or high gain regime the effective energy spread must be small compared to the intrinsic FEL efficiency η, $|\langle\delta\gamma_z\rangle|/\gamma_{z0} < \eta$. In the trapped particle regime, maintaining resonance implies that the effective energy spread must be small compared to the depth of the ponderomotive well, $|\langle\delta\gamma_z\rangle|/\gamma_{z0} < |e\Phi_p|/(\gamma mc^2)$, where Φ_p is the ponderomotive potential. For example, in the low-gain regime, $\eta = 1/(2N)$, where N is the number of wiggler periods. The inequality $|\langle\delta\gamma_z\rangle|/\gamma_{z0} < \eta$ implies $\delta\hat{B}_{rms} < 1/(\pi N) \simeq 0.3\%$ for $N = 100$ (where $a_w^2 \simeq 1$ has been assumed and $\delta\hat{B}_{rms} = \langle\delta\hat{B}_x\rangle^{1/2} = \langle\delta\hat{B}_y\rangle^{1/2}$). One should keep in mind that other sources exist which contribute to the effective energy spread (e.g., initial beam emittance and energy spreads due to transverse gradients in the wiggler fields) and these factors should be considered in a complete discussion of acceptable beam energy spreads in FELs.

IV. Deviations in the Relative Phase

To quantify how the parallel energy variation affects FEL gain, it is necessary to consider the relative phase ψ of the electrons in the ponderomotive wave,

$$d\psi/dz \equiv k + k_w - \omega/(c\beta_z). \tag{7}$$

As is discussed below, the FEL gain is directly determined by the behavior of the relative phase ψ. In the small signal limit ($a_R \to 0$, where a_R is the normalized radiation field), the electron energy is constant and the axial electron velocity is given by $\beta_z^2 = \beta_0^2 - \beta_\perp^2$, where β_0 is the initial normalized electron velocity. It is convenient to write $\beta_\perp = \beta_{\perp 0} + \delta\bar{\beta}_\perp$, where $\beta_{\perp 0}$ is the ideal electron wiggle velocity in the absence of field errors and $|\delta\bar{\beta}_\perp/\beta_{\perp 0}|^2 << 1$. The deviation in phase $\delta\psi$ due to the field errors[5-9] is given by

$$\delta\psi = -\frac{\omega}{2c}\beta_{z0}^{-3}\int_0^z dz'(2\beta_{\perp 0}\cdot\delta\bar{\beta}_\perp + \delta\bar{\beta}_\perp^2), \tag{8}$$

where $\beta_{z0}^2 = \beta_0^2 - \beta_{\perp 0}^2$. The specific behavior of $\delta\psi$ depends on the specific behavior of the transverse motion $\delta\bar{\beta}_\perp$ arising from the field errors. This motion has been examined in detail in Ref. 6 and is summarized in Appendix A. In the following, the mean phase deviation $\langle\delta\psi\rangle$ will be determined for (i) helical wigglers, (ii) linear wigglers with flat pole faces, (iii) linear wigglers with parabolic pole faces and (iv) an average of $\langle\delta\psi\rangle$ over a wiggler period will also be determined.

(i) Helical wigglers. Consider a helical wiggler (with weak focusing) described by the normalized vector potentials given by Eq. (A1) in Appendix A. The deviation in the transverse electron motion arising from the error $\delta a_\perp$ is given by

$$\delta\bar{\beta}_x = (a_w/2\gamma)k_w^2\delta y^2\cos k_w z + \delta\beta_x, \tag{9a}$$

$$\delta\bar{\beta}_y = (a_w/2\gamma)k_w^2\delta x^2\sin k_w z + \delta\beta_y, \tag{9b}$$

where the orbit deviations δx, δy, $\delta\beta_x$ and $\delta\beta_y$ are given by Eq. (A3)-(A6). Statistically averaging the phase deviation $\delta\psi$ over an ensemble of wigglers gives

$$\langle\delta\psi\rangle = -\frac{\omega}{2c}\left(\frac{a_w k_w}{\gamma}\right)^2\frac{z_c}{2}\left\{\left(\langle\delta\hat{B}_x^2\rangle + \langle\delta\hat{B}_y^2\rangle\right)z^2\right.$$

$$+\left(\langle\delta\hat{B}_x^2\rangle-\langle\delta\hat{B}_y^2\rangle\right)\left[\frac{z}{2k_w}\sin 2k_w z+\frac{1}{4k_w^2}(\cos 2k_w z-1)\right.$$
$$\left.+\frac{1}{4(k_w^2-k_\beta^2)}+\frac{1}{8k_\beta}\left(\frac{\cos 2(k_w+k_\beta)z}{(k_w+k_\beta)}-\frac{\cos 2(k_w-k_\beta)z}{(k_w-k_\beta)}\right)\right]\Bigg\}. \tag{10}$$

(ii) Flat pole faces. Consider a linear wiggler with flat pole faces described by the normalized vector potential given by Eq. (A11). The deviation in the transverse electron motion arising from the field errors is given by

$$\delta\bar{\beta}_x=(a_w/2\gamma)k_w^2\delta y^2\cos k_w z+\delta\beta_x, \tag{11a}$$

$$\delta\bar{\beta}_y=\delta\beta_y, \tag{11b}$$

where the orbit deviations δy, $\delta\beta_y$ and $\delta\beta_x$ are given by Eqs. (A4), (A6) and (A13). Statistically averaging the phase deviation $\delta\psi$ gives

$$\langle\delta\psi\rangle=-\frac{\omega}{2c}\left(\frac{a_w k_w}{\gamma}\right)^2\frac{z_c}{2}\Bigg\{\left(\langle\delta\hat{B}_x^2\rangle+\langle\delta\hat{B}_y^2\rangle\right)z^2$$
$$+\langle\delta\hat{B}_x^2\rangle\left[\frac{z}{2k_w}\sin 2k_w z+\frac{1}{4k_w^2}(\cos 2k_w z-1)\right.$$
$$\left.+\frac{1}{4(k_w^2-k_\beta^2)}+\frac{1}{8k_\beta}\left(\frac{\cos 2(k_w+k_\beta)z}{(k_w+k_\beta)}-\frac{\cos 2(k_w-k_\beta)z}{(k_w-k_\beta)}\right)\right]\Bigg\}. \tag{12}$$

(iii) Parabolic pole faces. Consider a linear wiggler with parabolic pole faces described by the normalized vector potential given by Eq. (A16). The deviation in the transverse electron motion arising from the field errors is given by

$$\delta\bar{\beta}_x=(a_w/4\gamma)k_w^2(\delta x^2+\delta y^2)\cos k_w z+\delta\beta_x, \tag{13a}$$

$$\delta\bar{\beta}_y=-(a_w/2\gamma)k_w^2\delta x\delta y\cos k_w z+\delta\beta_y, \tag{13b}$$

where the orbit deviations δx, δy, $\delta\beta_x$ and $\delta\beta_y$ are given by Eq. (A3)-(A6) with k_β replaced by $k_\beta/\sqrt{2}$. Statistically averaging the phase deviation $\delta\psi$ over an ensemble of wigglers gives

$$\langle\delta\psi\rangle=-\frac{\omega}{2c}\left(\frac{a_w k_w}{\gamma}\right)^2\frac{z_c}{2}\left(\langle\delta\hat{B}_x^2\rangle+\langle\delta\hat{B}_y^2\rangle\right)\Bigg\{z^2$$
$$+\frac{z}{2k_w}\sin 2k_w z+\frac{1}{4k_w^2}(\cos 2k_w z-1)+\frac{1}{2(2k_w^2-k_\beta^2)}$$
$$+\frac{1}{4k_\beta}\left[\frac{\cos(2k_w+\sqrt{2}k_\beta)z}{(\sqrt{2}k_w+k_\beta)}-\frac{\cos(2k_w-\sqrt{2}k_\beta)z}{(\sqrt{2}k_w-k_\beta)}\right]\Bigg\}. \tag{14}$$

(iv) Wiggler averaged result. Notice that if the above results for the mean phase deviation, Eqs. (10), (12) and (14), are averaged over a wiggler period, then to leading order $\langle\delta\psi\rangle$ is given by

$$\langle\delta\psi\rangle = -\frac{a_w^2 k_w^3}{2\gamma_\perp^2}\left(\langle\delta\hat{B}_x^2\rangle + \langle\delta\hat{B}_y^2\rangle\right) z_c z^2, \tag{15}$$

where the resonance condition $\omega/c = 2k_w\gamma^2/\gamma_\perp^2$ has been used and $z_c = z_{cx} = z_{cy}$ has been assumed. Here, $\gamma_\perp^2 = 1 + a_w^2$ for a helical wiggler and $\gamma_\perp^2 = 1 + a_w^2/2$ for a linear wiggler. Equation (15) is simply the result for the phase deviation as obtained from 1D theory in which transverse gradients (weak focusing) are neglected, i.e. $k_\beta = 0$. Hence, it is clear that transverse weak focusing (finite k_β) does not significantly reduce the mean phase deviation. (It should be mentioned tha in the trapped particle regime, the effects of the synchrotron motion of the electrons may reduce[4] $\langle\delta\psi\rangle$.)

Physically, $\delta\psi$ may be interpreted as an oscillation of the ponderomotive well due to field errors. Maintaining FEL resonance requires $\delta\psi$ to be small compared to 2π, i.e., the width of the well. In the low-gain regime, this phase deviation must be kept small over the entire wiggler length L. Requiring $|\langle\delta\psi(z = L)\rangle| << 2\pi$ implies $\delta\hat{B}_{rms} < 1/(\pi N) \sim 0.3\%$ for $N = 100$ (where $a_w^2 \simeq 1$ has been assumed and $\delta\hat{B}_{rms} = \langle\delta\hat{B}_x\rangle^{1/2} = \langle\delta\hat{B}_y\rangle^{1/2}$). This is the same condition as obtained above from considering the effective energy spread. In the high-gain regime,[7] the situation is somewhat different, since the length scale over which the FEL resonant interaction occurs is the e-folding length $1/\Gamma$, where Γ is the spatial growth rate of the radiation. Maintaining resonance in the high-gain regime corresponds to keeping $\delta\psi$ small over an e-folding length: $|\langle\delta\psi(z = 1/\Gamma)\rangle| < \pi$. Since, typically $1/\Gamma << L$, one expects the high-gain not to be strongly affected[7] by the phase deviation $\delta\psi$ (in contrast to the low-gain).

V. Degradation of FEL Gain

In principle, the magnitude of the random walk of the electron beam centroid may be kept as small as desired through the use of transverse focusing. This, however, is not the case with the phase deviation, as is discussed in the previous section. In the following, the effect of the phase deviation on the FEL gain in the low-gain regime is examined quantitatively. In determining the FEL gain, a number of assumptions are made. It is assumed that overlap is maintained between the radiation and electron beam, i.e., the random walk of the electron beam centroid remains smaller than the beam radius. Also, since weak focusing (transverse gradients) is ineffective in reducing the phase deviation, the gain will be considered in the 1D limit. The effects of tapering are neglected and a non-diffracting, plane wave radiation field is assumed. Furthermore, the effects of coupling to higher order harmonics (for linear wigglers) will be neglected. For a relativistic electron beam, the normalized amplitude gain, G, is related to the relative phase of the electrons, ψ, by[17]

$$G = \int_0^z dz' \left[\sin\psi\right]_0 , \tag{16}$$

where $[...]_0$ signifies an average over the intial phase of the electrons. The quantity G is proportional to the standard definition of the small signal gain.[17]

The relative phase ψ may be determined in the small-signal regime for which $a_w^2 >> \delta a_\perp^2 >> a_R^2$. The relative phase may be written as $\psi = \psi^{(0)} + \psi_1$, where $\psi^{(0)}$ is the relative phase in the absence of the radiation field,

$$\psi^{(0)}(z) = \psi_0 + \mu k_w z + \delta\psi(z), \tag{17}$$

and where ψ_1 is the phase contribution resulting from the radiation field. Here, ψ_0 is the initial phase of the electrons, μ is the normalized frequency mismatch,

$$\mu = -(\omega - \omega_0)/\omega_0, \tag{18}$$

where $\omega_0 = ck_w(1+\beta_{z0})\beta_{z0}\gamma^2/\gamma_\perp^2$ is the resonant frequency, and $\delta\psi$ is the phase deviation due to random field errors as is given by Eq. (8). The phase contribution resulting from the radiation field, ψ_1, is determined from the pendulum equation[17]

$$\frac{d^2}{dz^2}\psi_1 = \frac{4k_w^2}{\gamma_\perp^2} a_w a_R \sin\psi^{(0)}(z), \tag{19}$$

which gives

$$\psi_1(z) = \frac{4k_w^2}{\gamma_\perp^2} a_w a_R \int_0^z dz'(z - z') \sin\psi^{(0)}(z'). \tag{20}$$

Assuming $|\psi_1| << 1$, the expression for G may be expanded giving

$$G = \int_0^z dz' \left[\psi_1 \cos\psi^{(0)}\right]_0 . \tag{21}$$

Inserting the expression for ψ_1, Eq. (20), indicates that the normalized amplitude gain is proportional to $\hat{G}$, where

$$\hat{G} = \int_0^z dz' \int_0^{z'} dz'' \left[(z' - z'') \sin\left(\psi^{(0)}(z') - \psi^{(0)}(z'')\right)\right]_0 , \tag{22}$$

where $\left[\sin(\psi^{(0)}(z') + \psi^{(0)}(z''))\right]_0 = 0$ has been used. Averaging over an ensemble of wigglers gives an expression for the mean normalized gain, $\langle\hat{G}\rangle$,

$$\langle\hat{G}\rangle = \int_0^z dz' \int_0^{z'} dz'' \, (z' - z'') \, \langle \sin\left[\mu k_w \,(z' - z'') + \Delta\delta\psi\,(z', z'')\right]\rangle , \tag{23}$$

where $\Delta\delta\psi(z', z'') = \delta\psi(z') - \delta\psi(z'')$. When $\Delta\delta\psi = 0$, $\langle\hat{G}\rangle$ gives the normalized mean gain in the absence of field errors.

The precise evaluation of the ensemble average $\langle\hat{G}\rangle$ is nontrivial. In particular, the evaluation of $\langle\hat{G}\rangle$ is dependent upon the statistical distribution of the phase deviation $\delta\psi$. Consider random field errors $\delta a_\perp$ which are Gaussian distributed. If the terms linear in the field errors (terms proportional to $\delta\beta_\perp$) dominate the integrand in the expression for $\delta\psi$, Eq. (8), then $\delta\psi$ will tend to be Gaussian distributed. If the terms quadratic in the field errors (terms proportional to $\delta\beta_\perp^2$) dominate the integrand in the expression for $\delta\psi$, Eq. (8), then $\delta\psi$ will tend to be Gamma distributed. More specifically, the statistical behavior of $\delta\psi$ depends on whether the linear or quadratic terms dominate in the expression for variance σ of $\delta\psi$, where $\sigma^2 = \langle\delta\psi^2\rangle - \langle\delta\psi\rangle^2$. It may be shown (see Appendix B) that for long wigglers, in which the effects of field errors are important, σ is dominated by the quadratic terms in Eq. (8). Hence, the linear terms in Eq. (8) may be neglected and the phase deviation, in 1D, may be approximated by

$$\delta\psi = -(k_w/\gamma_\perp^2) \int_0^z dz' \delta a_\perp^2(z'). \tag{24}$$

Furthermore, the variable $\Delta\delta\psi$ is assumed to be Gamma distributed. Hence, the statistical average $\langle\hat{G}\rangle$ may be evaluated using the Rice–Mandel approximation,[2,18] giving

$$\begin{aligned}\langle\hat{G}\rangle = \int_0^z dz' \int_0^{z'} dz''(z'-z'')\left(1+\langle\Delta\delta\psi\rangle^2/f^2\right)^{-f/2} \\ \times \sin\left[\mu k_w(z'-z'') + f\tan^{-1}\left(\langle\Delta\delta\psi\rangle/f\right)\right],\end{aligned} \tag{25}$$

where $f = \langle\Delta\delta\psi\rangle^2/\left(\langle\Delta\delta\psi^2\rangle - \langle\Delta\delta\psi\rangle^2\right)$. A more detailed discussion of the statistical evaluation of $\langle\hat{G}\rangle$ is given in Appendix B. Equation (25) describes the reduction of the mean gain due to random fields errors in the low-gain regime for an untapered wiggler.

The mean of the quantity $\Delta\delta\psi$, as well as the square of its variance, may be calculated analytically. Using Eq. (24), one finds

$$\langle\Delta\delta\psi\rangle = -\frac{k_w^3 a_w^2}{2\gamma_\perp^2}\left(\langle\delta\hat{B}_x^2\rangle + \langle\delta\hat{B}_y^2\rangle\right) z_c\left(z'^2 - z''^2\right), \tag{26}$$

$$\begin{aligned}\langle\Delta\delta\psi^2\rangle - \langle\Delta\delta\psi\rangle^2 = \frac{k_w^6 a_w^4}{3\gamma_\perp^4}\left(\langle\delta\hat{B}_x^2\rangle + \langle\delta\hat{B}_y^2\rangle\right)^2 z_c^2(z'-z'')^2 \\ \times\left(z'^2 + 3z''^2 + 2z'z''\right),\end{aligned} \tag{27}$$

where $z_c = z_{cx} = z_{cy}$.

The mean gain $\langle\hat{G}\rangle$ (normalized to the maximum gain in the absence of field errors) is a function of only two parameters: the product of the frequency mismatch with the number of periods, μN, and the mean phase deviation at the wiggler end,

$$\langle\delta\psi\rangle_{max} = \langle\delta\psi(z=L)\rangle = -\frac{k_w^3 a_w^2}{2\gamma_\perp^2}\left(\langle\delta\hat{B}_x^2\rangle + \langle\delta\hat{B}_y^2\rangle\right) z_c L^2. \tag{28}$$

Using these two parameters, $\langle\hat{G}\rangle$ may be written as

$$\begin{aligned}\langle\hat{G}\rangle = L^4 \int_0^{\hat{z}} d\hat{z}' \int_0^{\hat{z}'} d\hat{z}''(\hat{z}'-\hat{z}'')\left(1+\langle\delta\psi\rangle_{max}^2 h^2\right)^{-f/2} \\ \times \sin\left[2\pi\mu N(\hat{z}'-\hat{z}'') + f\tan^{-1}\left(\langle\delta\psi\rangle_{max} h\right)\right],\end{aligned} \tag{29}$$

where $\hat{z} = z/L$ and

$$f = (3/4)\left(\hat{z}' + \hat{z}''\right)^2/\left(\hat{z}'^2 + 3\hat{z}''^2 + \hat{z}'\hat{z}''\right), \tag{30a}$$

$$h = \left(\hat{z}'^2 - \hat{z}''^2\right)/f. \tag{30b}$$

Equation (29) indicates that $\langle \hat{G} \rangle$ decreases as $\langle \delta\psi \rangle_{max}$ increases. In a similar fashion, it is possible to calculate an expression for the variance of the gain, the result of which is given in Appendix D. This variance tends to be large, as is indicated by the numerical simulations discussed below.

Equation (25) may be evaluated numerically to determine the behavior of the mean gain. Figure 1 illustrates this behavior, in which the mean gain $\langle \hat{G} \rangle$ is plotted as a function of the frequency mismatch parameter μN for several values of normalized rms field error $\delta\hat{B}_{rms}$ (0.0%, 0.1%,...,0.5%). The parameters in Fig. 1 correspond to a linearly polarized wiggler with $B_w = 5.4$ kG, $\lambda_w = 2.8$ cm, $N = 130$ and $\gamma = 350$ in the limit $k_\beta = 0$ (transverse focusing is neglected). Notice that with increasing rms field error, the maximum mean gain decreases and the position of this maximum moves to higher values of frequency mismatch. Figure 2 shows the peak gain $\langle \hat{G} \rangle_{max}$ as a function of normalized rms field error $\delta\hat{B}_{rms}$, as obtained from Eq. (25), for the above parameters. The $\times$'s in Fig. 2 are the result of an FEL simulation code for individual wiggler realizations (particular arrangements of random field errors). In these simulation runs, a random field error model similar to that of Kincaid[2,6-8] was used along with an electron beam of current 2.0 A with an emittance of 10 μm-rad. Notice that the large spread in the simulation results indicates a relatively large variance of the gain.

It is also possible to calculate the effect of wiggler errors on the spatial growth rate in the high-gain regime.[7] Numerical results (for a linear wiggler with $B_w = 2.4$ kG, $\lambda_w = 8.0$ cm and $L = 15$ m; and an electron beam with energy 50 MeV, current 1.5 kA and emittance 4.4 mm-mrad) indicate that even for large normalized rms field errors, $\delta\hat{B}_{rms} = 0.5\%$, the mean spatial growth rate is only slightly reduced (by $< 4\%$). This is in agreement with the discussion presented at the end of Section IV.

VI. Beam Steering

One method for reducing the detrimental effects of field errors is through the use of beam steering[2-10] (external fields are used to steer the electron beam back to axis). Analytically, this may be modeled by injecting the electron beam with an initial transverse velocity $\beta_{\perp i}$ such that the centroid displacement is zero at the end of the wiggler $\delta x(z = L) = 0$. The intial transverse velocity may be specified in terms of the perturbed transverse velocity in the absence of steering $\delta\beta_{\perp N}$ by the relation

$$\beta_{\perp i} = -\frac{1}{L}\int_0^L dz'\delta\beta_{\perp N}(z'), \tag{31}$$

where $\delta\beta_{\perp N}$ is given by Eq. (2). In the 1D limit, $\delta\beta_{\perp N}(z) = \delta a_{\perp}(z)/\gamma$.

Using the above expression for $\beta_{\perp i}$, one may calculate the electron motion in the presence of the field errors including the effects of beam steering. For example, the mean square transverse orbit deviation in the absence of transverse focusing ($k_\beta = 0$) is given by

$$\langle\delta x^2\rangle = \frac{a_w^2 k_w^2}{3\gamma^2}\langle\delta\hat{B}_y^2\rangle z_{cy}\left[z^3 + \epsilon z^2 L\left(1 - \frac{3z}{L} + \frac{z^2}{L^2}\right)\right], \tag{32}$$

where $\epsilon = 1$ with steering and $\epsilon = 0$ without steering, as derived previously by Kincaid.[2] Notice that with steering, the rms transverse orbit displacement is maximum at $z = L/2$ and is equal to 1/4 the value of the rms displacement obtained at the end of the wiggler in the absence of steering, i.e.,

$$\langle\delta x^2(z = L/2, \epsilon = 1)\rangle^{1/2} = (1/4)\langle\delta x^2(z = L, \epsilon = 0)\rangle^{1/2}. \tag{33}$$

Similarly, the phase deviation with ($\epsilon = 1$) and without ($\epsilon = 0$) steering in the absence of transverse focusing ($k_\beta = 0$) is given by

$$\langle\delta\psi\rangle = -\frac{a_w^2 k_w^3}{2\gamma_\perp^2}\left(\langle\delta\hat{B}_x^2\rangle + \langle\delta\hat{B}_y^2\rangle\right) z_c\left[z^2 + \epsilon\frac{2zL}{3}\left(1 - \frac{3z}{L} + \frac{z^2}{L^2}\right)\right], \tag{34}$$

where it has been assumed $z_c = z_{cx} = z_{cy}$. Both with and without steering, the mean phase deviation reaches a maximum at $z = L$. In particular, notice that the effect of steering is to reduce the mean phase deviation at the wiggler end by a factor of 1/3,

$$\langle\delta\psi(z = L, \epsilon = 1)\rangle = (1/3)\langle\delta\psi(z = L, \epsilon = 0)\rangle. \tag{35}$$

It is also possible to calculate the effect of steering on the mean gain, $\langle \hat{G} \rangle$. Again, Eq. (25) applies, where the mean and variance of the quantity $\Delta\delta\psi$ including the effects of steering are given in Appendix C. Similarly, an expression for the variance of the gain including steering is given in Appendix D.

The effect of beam steering at the wiggler entrance on the phase deviation $\delta\psi$ is illustrated in Fig. 3 for the cases (a) without steering and (b) with steering. Here the solid curves represent the mean $\langle \delta\psi \rangle$ and the dashed curves represent one standard deviation about the mean $\langle \delta\psi \rangle \pm \sigma$, where σ is the variance of the phase deviation. These plots are for a linearly polarized wiggler with $B_w = 5.4$ kG, $\lambda_w = 2.8$ cm, $N = 130$, $\gamma = 350$ and $\delta\hat{B}_{rms} = 0.3\%$ in the limit $k_\beta = 0$ (transverse focusing is neglected). Notice that the effect of steering at the wiggler entrance reduces $\langle \delta\psi \rangle$ by 1/3 at the end of the wiggler, as is indicated by Eq. (35). Also, notice that steering has reduced the variance of the phase deviation by an equally significant amount. For cases in which $k_\beta \neq 0$, it is possible to show[8] that steering reduces the mean phase deviation when the length over which the steering in performed is less than the betatron wavelength, $L_s < \lambda_\beta$. For cases in which $L_s > \lambda_\beta$, beam steering may increase the value of $\langle \delta\psi \rangle$.

The effect of beam steering at the wiggler entrance on the FEL gain (in the low-gain regime) is illustrated in Figs. 4-6. In Fig. 4, the mean gain $\langle \hat{G} \rangle$ including the effects of steering is plotted as a function of the frequency mismatch parameter μN for several values of normalized rms field error $\delta\hat{B}_{rms}$ (0.0%, 0.1%,...,0.5%). The parameters in Fig. 4 correspond to a linearly polarized wiggler with $B_w = 5.4$ kG, $\lambda_w = 2.8$ cm, $N = 130$ and $\gamma = 350$ in the limit $k_\beta = 0$ (transverse focusing is neglected). Figure 4, in comparison to Fig. 1, clearly indicates that the mean gain is enhanced through the use of steering. For example, for $\delta\hat{B}_{rms} = 0.3\%$, steering increases the peak gain by a factor of approximately 2.5. Figure 5 illustrates this comparison, in which the peak gain $\langle \hat{G} \rangle_{max}$, with and without the effects of steering, is plotted as a function of normalized rms field error $\delta\hat{B}_{rms}$ for the above parameters. In Fig. 6, the peak normalized gain $\langle \hat{G} \rangle_{max}$ is plotted as a function of the maximum mean phase deviation, $\langle \delta\psi \rangle_{max}$, for the above parameters without steering in Fig. 6(a) and with steering in Fig. 6(b). The curves in Fig. 6 remain unchanged for various values of $\delta\hat{B}_{rms}$ and N, hence, the maximum normalized mean gain $\langle\hat{G}\rangle_{max}$ is a function of only $\langle\delta\psi\rangle_{max}$. To avoid significant reductions in the mean gain, Fig. 6 indicates that it is necessary to have $|\langle\delta\psi\rangle| << 2\pi$. Figure 7 shows the peak mean gain $\langle\hat{G}\rangle_{max}$, including the effects of steering, plotted as a function of the normalized rms field error $\delta\hat{B}_{rms}$ for the above parameters. Included in Fig. 7 is the variance of the normalized gain, as obtained from Appendix D, for several values of $\delta\hat{B}_{rms}$. As was indicated by the kinetic simulations for individual wiggler realizations, the variance of the gain tends to be large and increases with increasing rms field error.

VII. Error Reduction Techniques

Several methods exist for reducing the detrimental effects of wiggler errors. Above it was discussed how steering[2–10] the electron beam at the entrance of the wiggler may improve FEL performance. This concept may be generalized to the case of multiple beam steering,[3,4,8,10,16] in which the electron beam is steered back to axis in several places along the length of the wiggler. It may be shown (see, for example, Ref. 8) that in order to reduce the phase deviation, it is necessary to perform steering over segments of length L_S shorter than a betatron wavelength, $L_S < \lambda_\beta$. In addition to beam steering, one may consider wiggler errors which are correlated.[8] The results discussed above are for wigglers with random errors which are assumed to be uncorrelated for separation distances greater than $z_c \simeq \lambda_w/2$. By considering a wiggler in which the error for a given magnet pole is correlated to the errors of the surrounding poles, one may construct beneficial correlations which reduce the detrimental effects of the errors.[8]

Alternatively, one may reduce the detrimental effects of the errors by considering an optimal arrangement of the magnet poles.[12–15] That is, the magnet poles are to be arranged in such a way that the detrimental effects of the error of a given pole tend to cancel those of the surrounding poles. More specifically, the magnet poles are arranged in such a way as to minimize an appropriate "cost function". For example, one may choose to arrange the poles such that the magnitude of random walk $|\delta x|$ is minimized, where $\delta x \sim \int dz' \sin k_\beta(z'-z)\delta\hat{B}_y(z')$. (Notice that minimization of $|\int dz \delta B|$ does not correspond to minimization of $|\delta x|$.) However, the results discussed above indicate that a more appropriate cost function for low-gain FELs is the magnitude of the phase deviation $|\delta\psi|$, $\delta\psi \sim \int dz'(2\beta_{\perp 0}\delta\bar{\beta}_\perp + \delta\bar{\beta}_\perp^2)$, as is given by Eq. (8). By minimizing $|\delta\psi|$, one reduces the amount of gain loss. Ideally, one would like to maximize the actual expression for the gain, Eq. (22), but the functional dependence of the gain on the field errors appears much too complicated to be of practical usefulness.

VIII. Conclusions

An electron beam traveling through a magnetic wiggler with finite field errors experiences random $v_z \times \delta B_\perp$ forces which perturbs the beam motion. This leads to a random walk of the beam centroid, δx, as well as a random deviation in the relative phase of the electrons in the ponderomotive wave, $\delta\psi$. In principle, the transverse displacement of the beam centroid, δx, may be kept as small as desired through the combined effects of transverse beam focusing and beam steering. This, however, is not the case for the phase deviation $\delta\psi$. Transverse beam focusing is found to be ineffective in reducing the mean phase deviation ($\langle\delta\psi\rangle$ is independent of k_β). Beam steering[8] may be used to reduce $|\delta\psi|$ only when $L_S < \lambda_\beta$. As an example, for the case $k_\beta = 0$ and using steering at the wiggler entrance indicates that the mean phase deviation at the wiggler end is reduced by a factor of 1/3. The phase deviation leads to a reduction of FEL gain (the low-gain regime is affected more strongly than the high-gain regime). The normalized mean gain was calculated and found to be a function of only two parameters, μN and $\langle\delta\psi\rangle_{max}$, as indicated by Eq. (29). To avoid significant loss of gain in the low-gain regime, it is desirable to keep $|\delta\psi| << 2\pi$. In particular, requiring $|\langle\delta\psi\rangle| << 2\pi$ gives, using Eq. (15),

$$\left(\langle\delta\hat{B}_x^2\rangle + \langle\delta\hat{B}_x^2\rangle\right)^{1/2} < \alpha/(\pi N), \tag{36}$$

where $\alpha = (1 + a_w^2)^{1/2}/a_w$ for a helical wiggler and $\alpha = (1 + a_w^2/2)^{1/2}/a_w$ for a planar wiggler. For example, a helical wiggler with $N = 100$, $a_w \simeq 1$ and $\delta\hat{B}_{rms} = \langle\delta\hat{B}_x\rangle^{1/2} = \langle\delta\hat{B}_y\rangle^{1/2}$ implies that the normalized rms field error must satisfy $\delta\hat{B}_{rms} < 0.3\%$. Possible error reduction techniques include multiple beam steering,[8,10] correlation of field errors[8] and optimal arrangement of magnet poles.[13-15] To reduce the detrimental effects of field errors on the gain in low-gain FELs, an optimal arrangement of poles corresponds to minimization of $|\delta\psi|$, where $\delta\psi$ is given by Eq. (8).

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Appendix A:
Transverse Orbit Deviations

This appendix summarizes the results of Ref. 6 in which the transverse orbit deviations arising from random wiggler field errors were calculated for (i) helical wigglers, (ii) linear wigglers with flat pole faces and (iii) linear wigglers with parabolic pole faces.

(i) Helical Wigglers. Consider a helical wiggler (with weak focusing) described by the normalized vector potential

$$\begin{aligned} a_x &= a_w(1 + k_w^2 y^2/2)\cos k_w z + \delta a_x(z), \\ a_y &= a_w(1 + k_w^2 x^2/2)\sin k_w z + \delta a_y(z), \end{aligned} \tag{A1}$$

where it is assumed $k_w^2 x^2 << 1$ and $k_w^2 y^2 << 1$. Here, δa_x and δa_y are related to the field errors δB_x and δB_y by

$$\begin{aligned} \delta a_x(z) &= a_w k_w \int_0^z dz' \delta B_y(z')/B_w, \\ \delta a_y(z) &= -a_w k_w \int_0^z dz' \delta B_x(z')/B_w. \end{aligned} \tag{A2}$$

The deviations in the transverse electron orbit arising from the errors $\delta a_\perp$ are given by

$$\delta x(z) = -\frac{a_w k_w}{\gamma k_\beta}\int_0^z dz' \sin k_\beta(z' - z)\frac{\delta B_y(z')}{B_w}, \tag{A3}$$

$$\delta y(z) = \frac{a_w k_w}{\gamma k_\beta}\int_0^z dz' \sin k_\beta(z' - z)\frac{\delta B_x(z')}{B_w}, \tag{A4}$$

which correspond to the normalized transverse velocity deviations

$$\delta\beta_x(z) = \frac{a_w k_w}{\gamma}\int_0^z dz' \cos k_\beta(z' - z)\frac{\delta B_y(z')}{B_w}, \tag{A5}$$

$$\delta\beta_y(z) = -\frac{a_w k_w}{\gamma}\int_0^z dz' \cos k_\beta(z' - z)\frac{\delta B_x(z')}{B_w}, \tag{A6}$$

where $k_\beta \equiv a_w k_w/\sqrt{2}\gamma$. Statistically averaging over an ensemble of wigglers gives the mean square quantities

$$\langle \delta x^2 \rangle = \left(\frac{a_w k_w}{\gamma k_\beta}\right)^2 \left\langle \frac{\delta B_y^2}{B_w^2} \right\rangle \frac{z_{cy}}{2}\left(z - \frac{\sin 2k_\beta z}{2k_\beta}\right), \tag{A7}$$

$$\langle \delta y^2 \rangle = \left(\frac{a_w k_w}{\gamma k_\beta}\right)^2 \left\langle \frac{\delta B_x^2}{B_w^2} \right\rangle \frac{z_{cx}}{2} \left(z - \frac{\sin 2k_\beta z}{2k_\beta}\right), \tag{A8}$$

$$\langle \delta\beta_x^2 \rangle = \left(\frac{a_w k_w}{\gamma}\right)^2 \left\langle \frac{\delta B_y^2}{B_w^2} \right\rangle \frac{z_{cy}}{2} \left(z + \frac{\sin 2k_\beta z}{2k_\beta}\right), \tag{A9}$$

$$\langle \delta\beta_y^2 \rangle = \left(\frac{a_w k_w}{\gamma}\right)^2 \left\langle \frac{\delta B_x^2}{B_w^2} \right\rangle \frac{z_{cx}}{2} \left(z + \frac{\sin 2k_\beta z}{2k_\beta}\right). \tag{A10}$$

(ii) Flat pole faces. Consider a linear wiggler with flat pole faces described by the normalized vector potential

$$\mathbf{a} = a_w \cosh k_w y \cos k_w z \mathbf{e}_x + \delta a_x(z)\mathbf{e}_x + \delta a_y(z)\mathbf{e}_y. \tag{A11}$$

The x component of the orbit deviation is described by

$$\delta x(z) = \frac{a_w k_w}{\gamma} \int_0^z dz' \int_0^{z'} dz'' \frac{\delta B_y(z'')}{B_w}, \tag{A12}$$

$$\delta\beta_x(z) = \frac{a_w k_w}{\gamma} \int_0^z dz' \frac{\delta B_y(z')}{B_w}, \tag{A13}$$

and the y component of the orbit deviation is described by Eqs. (A4) and (A6). Statistically averaging over an ensemble of wigglers gives the mean square quantities

$$\langle \delta x^2 \rangle = \left(\frac{a_w k_w}{\gamma}\right)^2 \left\langle \frac{\delta B_y^2}{B_w^2} \right\rangle z_{cy} \frac{z^3}{3}, \tag{A14}$$

$$\langle \delta\beta_x^2 \rangle = \left(\frac{a_w k_w}{\gamma}\right)^2 \left\langle \frac{\delta B_y^2}{B_w^2} \right\rangle z_{cy} z, \tag{A15}$$

where $\langle \delta y^2 \rangle$ and $\langle \delta\beta_y^2 \rangle$ are given by Eqs. (A8) and (A10).

(iii) Parabolic Pole Faces. Consider a linear wiggler with parabolic pole faces described by the normalized vector potential

$$\begin{aligned} a_x &= a_w \cosh(k_w x/\sqrt{2}) \cosh(k_w y/\sqrt{2}) \cos k_w z + \delta a_x(z), \\ a_y &= -a_w \sinh(k_w x/\sqrt{2}) \sinh(k_w y/\sqrt{2}) \cos k_w z + \delta a_y(z). \end{aligned} \tag{A16}$$

The deviations in the transverse electron orbit are given by Eqs. (A3)-(A6) with k_β replaced by $k_\beta/\sqrt{2}$. Similarly, the mean square quantities are given by Eqs. (A7)-(A10) with k_β replaced by $k_\beta/\sqrt{2}$.

Appendix B:
The Rice–Mandel Approximation

Evaluation of the average $\langle \hat{G} \rangle$, given by Eq. (23), is dependent upon the statistical distribution of the phase deviation $\delta\psi$. The expression for $\delta\psi$ consists of terms linear in the field error $\delta a_\perp$ as well as terms quadratic in the field error. The statistical behavior of $\delta\psi$ depends on whether the linear or quadratic terms dominate in the expression for the variance σ of $\delta\psi$, where $\sigma^2 = \langle \delta\psi^2 \rangle - \langle \delta\psi \rangle^2$. It is possible to calculate the contribution of the linear terms to the variance of the phase deviation, σ_L, as well as the contribution of the quadratic terms to the variance, σ_Q, in the 1D limit with or without the effects of steering. One finds that the relative magnitude of quadratic terms to the linear terms in the phase variance is given by

$$\frac{\sigma_Q^2}{\sigma_L^2} \simeq \frac{2\pi^4}{3} \langle \delta \hat{B}_\perp^2 \rangle N^3 g(\hat{z}), \tag{B1}$$

where $N = L/\lambda_w$ is the number of wiggler periods, L is the wiggler length, $\hat{z} = z/L$ and $z_c \simeq \lambda_w/2$ was assumed. In the absence of steering, $g(\hat{z}) \simeq \hat{z}^3$, whereas with beam steering at the wiggler entrance (one steering segment), $g(\hat{z})$ is given by

$$g(\hat{z}) \simeq 4\hat{z}^2(1 + 3\hat{z}^2)^{-1} \left[\left(2 - 8\hat{z} + 13\hat{z}^2\right) + 0.2\left(-48\hat{z}^3 + 14\hat{z}^4\right) \right]. \tag{B2}$$

With steering, Eq. (B2) gives $g(1/4) \simeq 0.14$, $g(1/2) \simeq 0.13$, $g(3/4) \simeq 0.12$ and $g(1) \simeq 0.20$. As an example, consider $\langle \delta \hat{B}_\perp^2 \rangle \simeq 0.3\%$ and $g(\hat{z}) \simeq 0.1$. Require $\sigma_Q^2/\sigma_L^2 >> 1$ indicates that $N >> 26$. Hence, for long wigglers the statistical behavior of $\delta\psi$ is dominated by the quadratic terms and the linear terms may be neglected in Eq. (8).

If the linear terms are neglected in Eq. (8), then the equation for the phase deviation reduces to the generic form

$$y = \int_0^z dz' x^2(z'), \tag{B3}$$

where y represents the phase deviation and x represents the random field error. If x is Gaussian distributed with zero mean, then y will tend to obey a Gamma distribution. The Rice–Mandel approximation[18] assumes that the probability distribution for y, $P(y)$ has the general form of a Gamma distribution. The parameters occurring in this general form are determined by moment matching, i.e., by requiring $\int dyP(y) = 1$, $\int dyP(y)y = \langle y\rangle$ and $\int dyP(y)y^2 = \sigma_y^2 + \langle y\rangle^2$, where the mean $\langle y\rangle$ and the variance σ_y are assumed to be known. One finds

$$P(y) = \frac{1}{\Gamma(s)}\left(\frac{s}{\langle y\rangle}\right)^s y^{s-1}\exp\left(-\frac{sy}{\langle y\rangle}\right), \tag{B4}$$

where $s \equiv \langle y\rangle^2/\sigma_y^2$. Knowing the distribution $P(y)$ enables various statistical averages to be calculated. For example,

$$\langle \exp(iy)\rangle \equiv \int_0^\infty dyP(y)\exp(iy) = \left(1+\frac{\langle y\rangle^2}{s^2}\right)^{-s/2}\exp\left(is\tan^{-1}\frac{\langle y\rangle}{s}\right). \tag{B5}$$

Notice that $\langle y\rangle$ decreases algebraically as $\langle y\rangle^2$ increases (assuming s to be roughly constant). The imaginary part of the above expression was used to calculate $\langle\hat{G}\rangle$ in Eq. (25).

As a final note, it should be mentioned that if the linear terms in the expression for the phase deviation dominate the statistical behavior, then $y = \int dz'x(z')$ would be Gaussian distributed. In particular, $\langle\exp(iy)\rangle = \exp(i\langle y\rangle - \sigma_y^2/2)$. Hence, if y is Gaussian distributed, $\langle\exp(iy)\rangle$ decreases exponentially as σ_y^2 increases. This implies that for Gaussian distributed phase deviations $\delta\psi$, the mean gain $\langle\hat{G}\rangle$ would decrease much more rapidly with increasing $\langle\delta\psi\rangle^2$ than is predicted by the Rice–Mandel approximation, Eq. (25).

Appendix C:
Statistical Moments of $\Delta\delta\psi$

Various statistical moments of the function $\Delta\delta\psi = \delta\psi(z') - \delta\psi(z'')$ may be calculated analytically in the 1D limit including the effects of beam steering at wiggler entrance (one steering segment). In particular, the mean $\langle\Delta\delta\psi\rangle$ and the square of the variance $\sigma_\Delta^2 = \langle\Delta\delta\psi^2\rangle - \langle\Delta\delta\psi\rangle^2$ may be calculated using Eqs. (8) and (31). One finds

$$\begin{aligned}\langle\Delta\delta\psi\rangle = &-\frac{k_w^3 a_w^2}{\gamma_\perp^2}\langle\delta\hat{B}_\perp^2\rangle z_c\,(z'-z'')\\ &\times\left\{\frac{1}{2}(z'+z'')+\epsilon\left[\frac{L}{3}-(z'+z'')+\frac{1}{3L}\left(z'^2+z''^2+z'z''\right)\right]\right\},\end{aligned} \tag{C1}$$

$$\begin{aligned}\sigma_\Delta^2 = &\frac{k_w^6 a_w^4}{\gamma_\perp^4}\langle\delta\hat{B}_\perp^2\rangle^2 z_c^2\,(z'-z'')^2\\ &\times\left\{\frac{1}{3}\left(z'^2+3z''^2+2z'z''\right)+\epsilon\left[\frac{2}{9}\left(5z'^2-z''^2+2z'z''\right)\right.\right.\\ &-\frac{2L}{9}(4z'+2z''-L)-\frac{1}{15L}\left(16z'^3+9z''^3+17z'^2z''+18z''^2z'\right)\\ &\left.\left.+\frac{1}{45L^2}\left(14\left(z'^4+z''^4\right)+19z'z''\left(z'^2+z''^2\right)+24z'^2z''^2\right)\right]\right\},\end{aligned} \tag{C2}$$

where $\epsilon = 0$ without steering and $\epsilon = 1$ with steering. These expressions may be used to evaluate the mean gain with steering using Eq. (25), where $f = \langle\Delta\delta\psi\rangle^2/\sigma_\Delta^2$.

Appendix D:
Gain Variance

It is possible to calculate an approximate expression for the variance of the normalized gain σ_G, where

$$\sigma_g^2 = \langle \hat{G}^2 \rangle - \langle \hat{G} \rangle^2. \tag{D1}$$

This expression is given here for completeness. The mean square normalized gain is given by

$$\langle \hat{G}^2 \rangle = I_1 + I_2 + I_3 + I_4 + I_5, \tag{D2}$$

where

$$I_1 = \int_0^z dz_1 \int_0^{z_1} dz_2 \int_0^{z_1} dz_1' \int_0^{z_1'} dz_2' \frac{1}{2} \Delta z_1 \Delta z_2 F(z_1, z_1, z_2, z_2'), \tag{D3}$$

$$I_2 = \int_0^z dz_1 \int_{z_1}^{z} dz_2 \int_0^{z_1} dz_1' \int_{z_1'}^{z_2} dz_2' \frac{1}{2} \Delta z_1 \Delta z_2 F(z_1, z_1', z_2, z_2'), \tag{D4}$$

$$I_3 = \int_0^z dz_1 \int_0^{z_1} dz_2 \int_0^{z_1} dz_1' \int_{z_1'}^{z_2} dz_2' \frac{1}{2} \Delta z_1 \Delta z_2 H_{(-)}(z_1, z_1', z_2, z_2'), \tag{D5}$$

$$I_4 = \int_0^z dz_1 \int_{z_1}^{z} dz_2 \int_0^{z_1} dz_1' \int_0^{z_1'} dz_2' \frac{1}{2} \Delta z_1 \Delta z_2 H_{(-)}(z_1, z_1', z_2, z_2'), \tag{D6}$$

$$I_5 = -\int_0^z dz_1 \int_0^{z} dz_2 \int_0^{z_1} dz_1' \int_0^{z_2} dz_2' \frac{1}{2} \Delta z_1 \Delta z_2 H_{(+)}(z_1, z_1', z_2, z_2'), \tag{D7}$$

where $\Delta z_1 = z_1 - z_1'$ and $\Delta z_2 = z_2 - z_2'$. The functions F and $H_{(\pm)}$ are given by

$$F = \exp\left(-\frac{1}{2}\sigma_{\Delta(-)}^2\right) \cos\left[\mu k_w(\Delta z_1 - \Delta z_2) + \langle \Delta(-) \rangle\right], \tag{D8}$$

$$\begin{aligned} H_{(\pm)} = &\left(1 + \frac{\langle \Delta(\pm) \rangle^2}{f_{\Delta(\pm)}^2}\right)^{-f_{\Delta(\pm)}/2} \\ &\times \cos\left[\mu k_w(\Delta z_1 \pm \Delta z_2) + f_{\Delta(\pm)} \tan^{-1}\left(\frac{\langle \Delta(\pm) \rangle}{f_{\Delta(\pm)}}\right)\right], \end{aligned} \tag{D9}$$

where $f_{\Delta(\pm)} = \langle \Delta(\pm) \rangle^2 / \sigma_{\Delta(\pm)}^2$ and

$$\langle \Delta(\pm) \rangle = \langle \Delta(z_1, z_1') \rangle \pm \langle \Delta(z_2, z_2') \rangle, \tag{D10}$$

$$\begin{aligned} \sigma_{\Delta(\pm)}^2 = &\sigma_{\Delta(z_1,z_1')}^2 + \sigma_{\Delta(z_2,z_2')}^2 \\ &\pm \left(\sigma_{\Delta(z_1,z_2')}^2 + \sigma_{\Delta(z_1',z_2)}^2 - \sigma_{\Delta(z_1,z_2)}^2 - \sigma_{\Delta(z_1',z_2')}^2\right), \end{aligned} \tag{D11}$$

with

$$\langle\Delta(x,y)\rangle = -\frac{k_w^3 a_w^2}{\gamma_\perp^2}\langle\delta\hat{B}_\perp^2\rangle z_c\,(x-y) \times \left\{\frac{1}{2}(x+y)+\epsilon\left[\frac{L}{3}-(x+y)+\frac{1}{3L}\left(x^2+y^2+xy\right)\right]\right\}, \tag{D12}$$

$$\begin{aligned}\sigma^2_{\Delta(x,y)} = &\frac{k_w^6 a_w^4}{\gamma_\perp^4}\langle\delta\hat{B}_\perp^2\rangle^2 z_c^2\,(x-y)^2 \\ &\times\left\{\frac{1}{3}\left(x^2+3y^2+2xy\right)+\epsilon\left[\frac{2}{9}\left(5x^2-y^2+2xy\right)\right.\right. \\ &-\frac{2L}{9}(4x+2y-L)-\frac{1}{15L}\left(16x^3+9y^3+17x^2y+18y^2x\right) \\ &\left.\left.+\frac{1}{45L^2}\left(14\left(x^4+y^4\right)+19xy\left(x^2+y^2\right)+24x^2y^2\right)\right]\right\},\end{aligned} \tag{D13}$$

where $\epsilon = 0$ without steering and $\epsilon = 1$ with steering. Equation (D13) only holds for $x > y$. When $x < y$, the variables x and y need to be interchanged in Eq. (D13). These expressions, along with the expression for the mean gain, Eq. (25), may be used to evaluate the gain variance, σ_G, with or without the effects of steering. Numerically, this evaluation is nontrivial.

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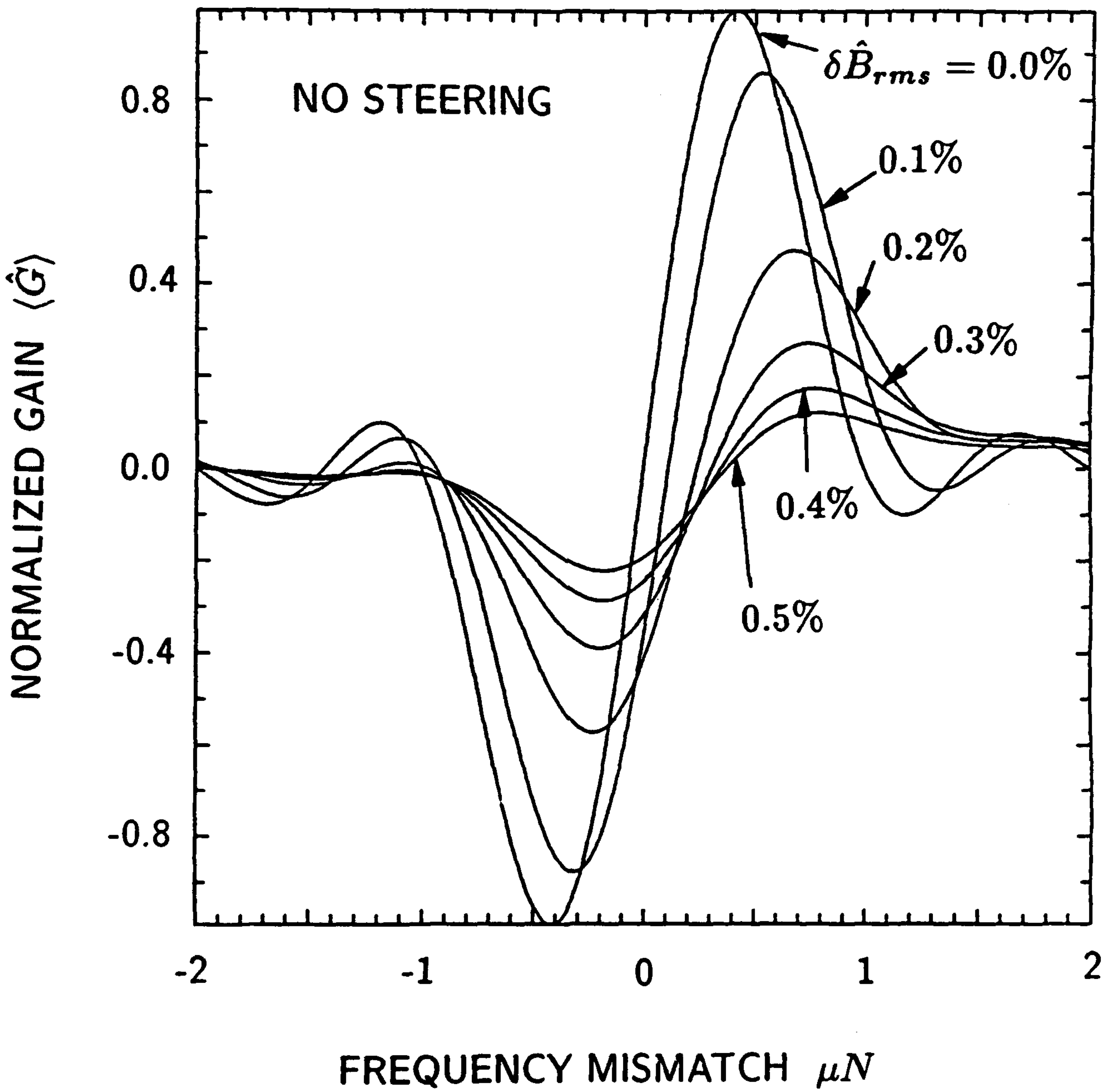

Fig. 1. Mean gain $\langle\hat{G}\rangle$ versus frequency mismatch μN for several values of rms field error $\delta\hat{B}_{rms}$ (0.0%, 0.1%,...,0.5%) for a linearly polarized wiggler with B_w = 5.4 kG, λ_w = 2.8 cm, N = 130 and γ = 350 in the limit k_β = 0.

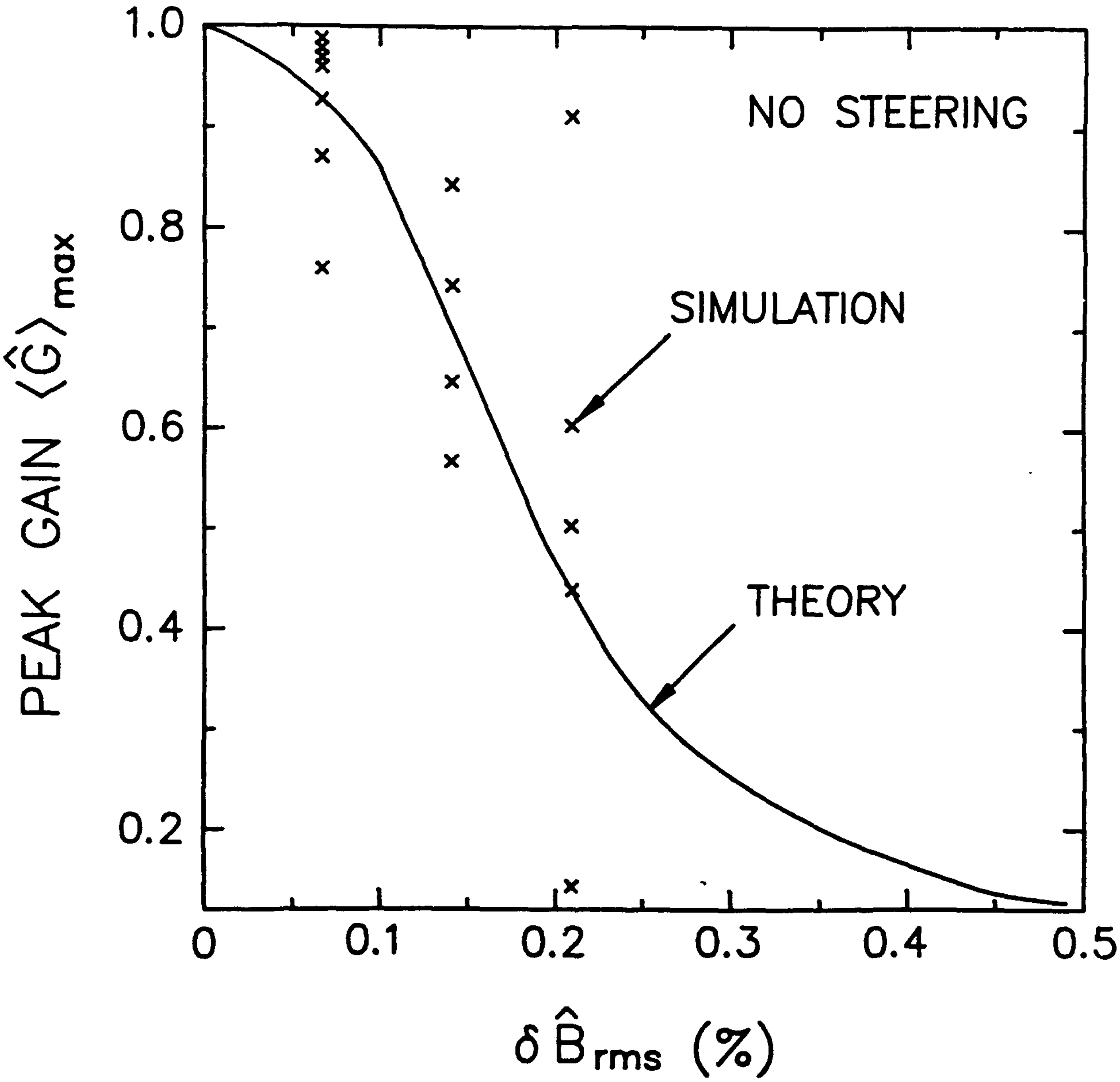

Fig. 2. Peak mean gain $\langle\hat{G}\rangle_{max}$ versus normalized rms field error $\delta\hat{B}_{rms}$ for a linearly polarized wiggler with B_w = 5.4 kG, λ_w = 2.8 cm, N = 130 and γ = 350 in the limit $k_\beta = 0$. The solid curve denotes the theoretical result and the $\times$'s denote FEL simulations for individual wiggler realizations.

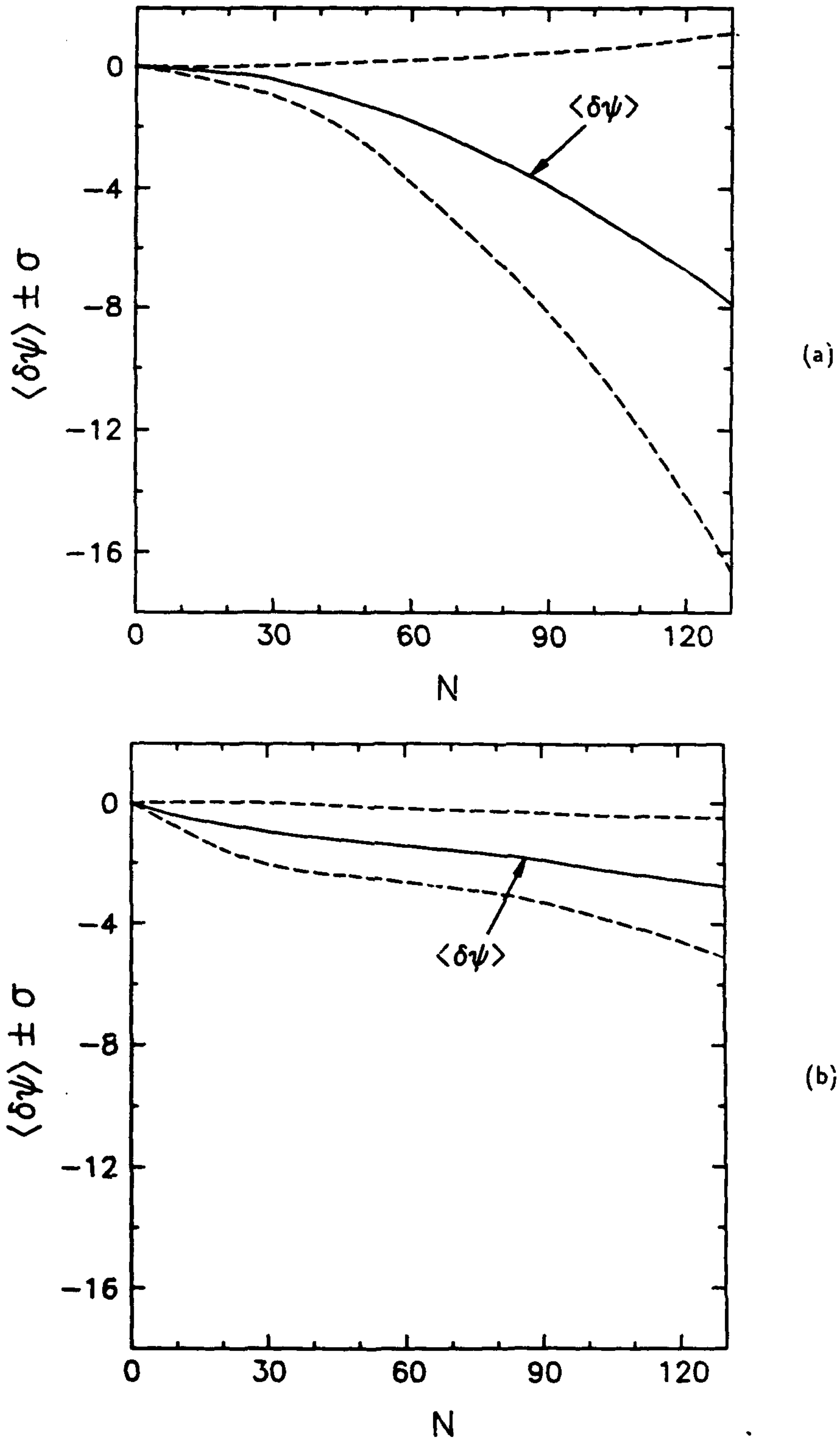

Fig. 3. Phase deviation $\delta\psi$ versus number of wiggler periods N (a) without steering and (b) with steering for a linearly polarized wiggler with $B_w = 5.4$ kG, $\lambda_w = 2.8$ cm, $\gamma = 350$ and $\delta\hat{B}_{rms} = 0.3\%$ in the limit $k_\beta = 0$. The solid curves represent the mean $\langle\delta\psi\rangle$ and the dashed curves represent one standard deviation σ about the mean.

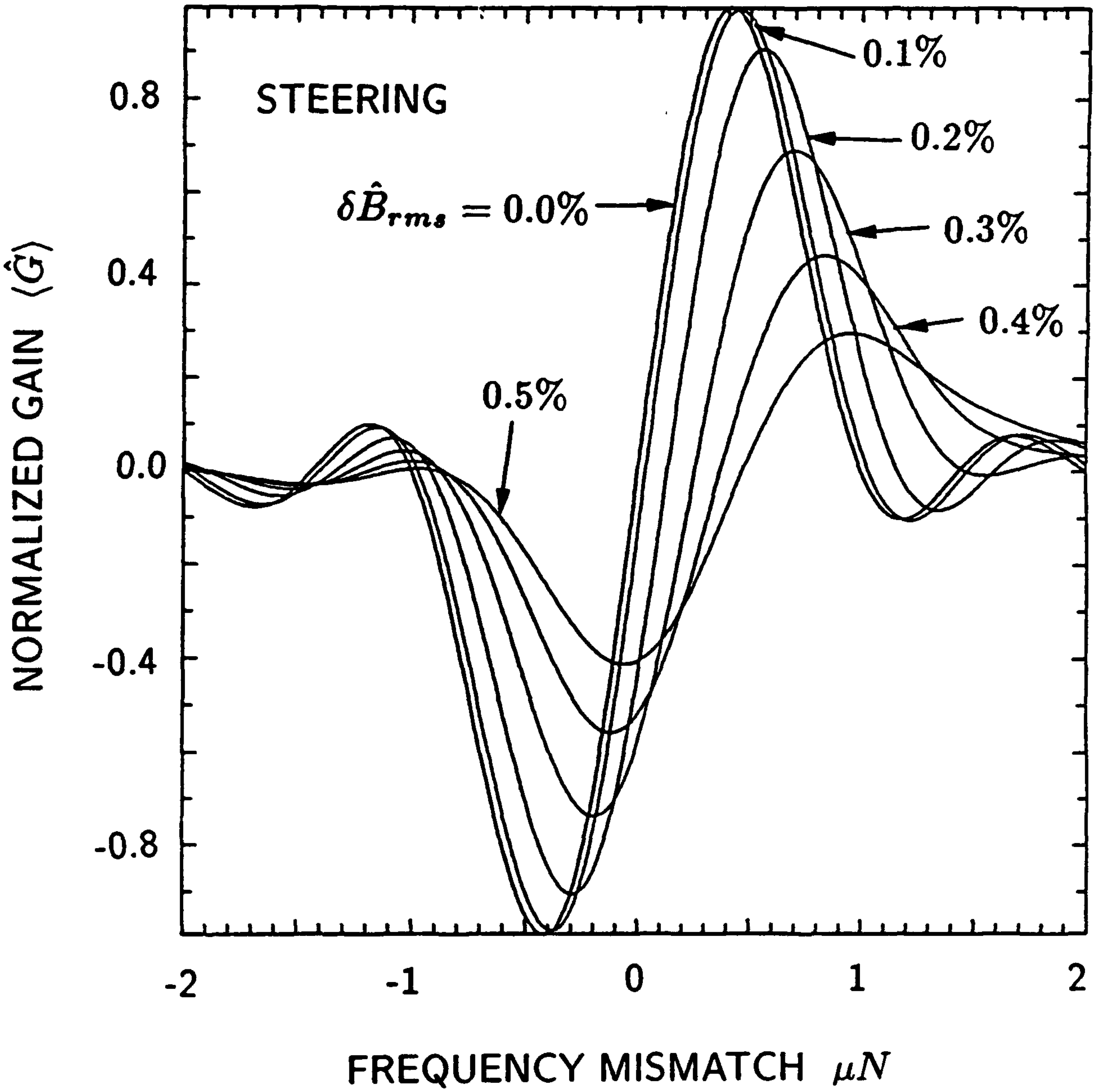

Fig. 4. Mean gain $\langle \hat{G} \rangle$ including the effects of steering versus frequency mismatch μN for several values of rms field error $\delta \hat{B}_{rms}$ (0.0%, 0.1%,...,0.5%) for a linearly polarized wiggler with $B_w = 5.4$ kG, $\lambda_w = 2.8$ cm, $N = 130$ and $\gamma = 350$ in the limit $k_\beta = 0$.

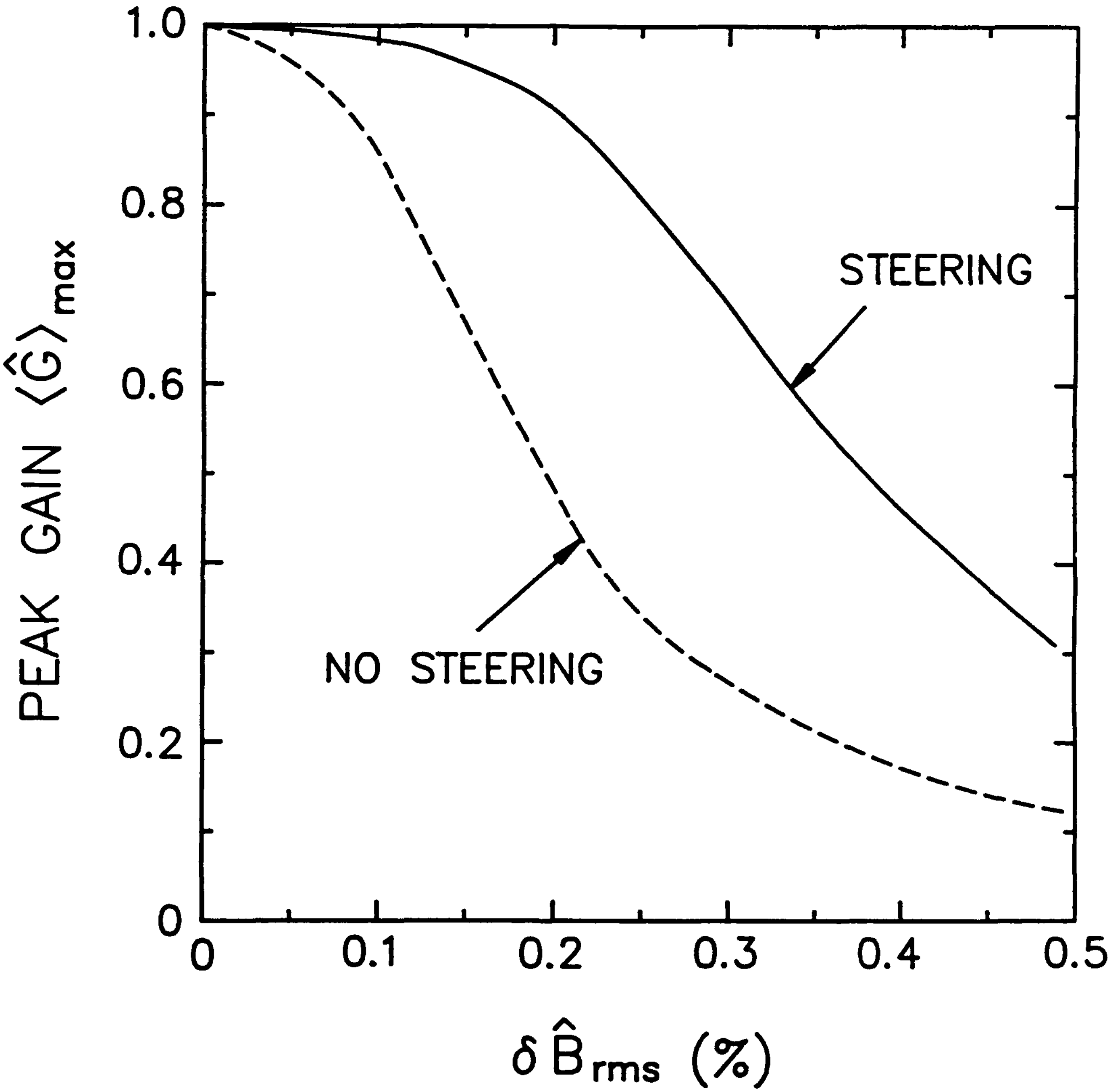

Fig. 5. Peak gain $\langle\hat{G}\rangle_{max}$ versus normalized rms field error $\delta\hat{B}_{rms}$, with steering (solid curve) and without steering (dashed curve) for a linearly polarized wiggler with $B_w = 5.4$ kG, $\lambda_w = 2.8$ cm, $N = 130$ and $\gamma = 350$ in the limit $k_\beta = 0$.

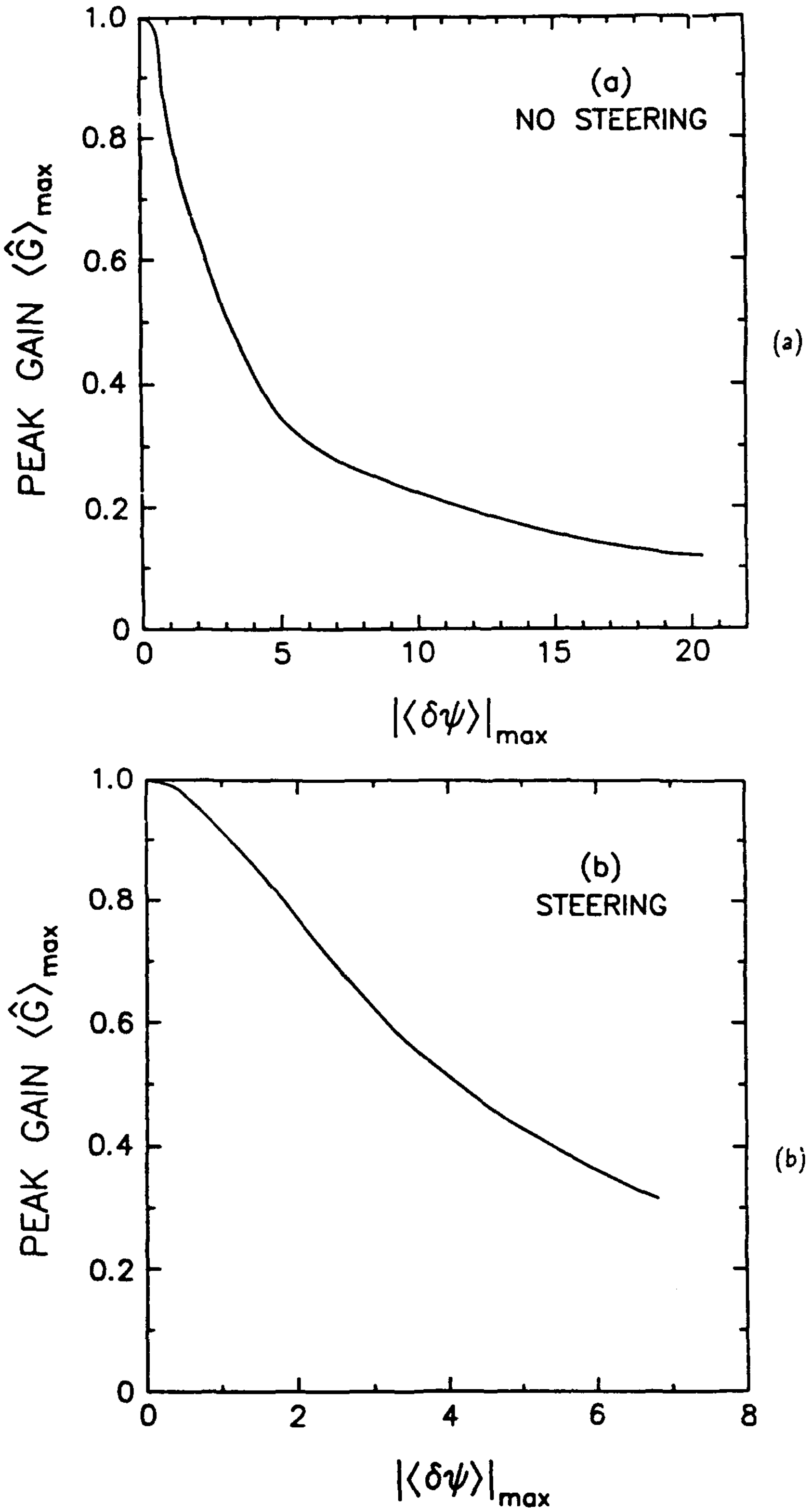

Fig. 6. Peak gain $\langle \hat{G} \rangle_{max}$ versus maximum phase deviation $\langle \delta\psi \rangle_{max}$ for a linearly polarized wiggler with $B_w = 5.4$ kG, $\lambda_w = 2.8$ cm and $\gamma = 350$ in the limit $k_\beta = 0$ for (a) no steering and (b) with steering.

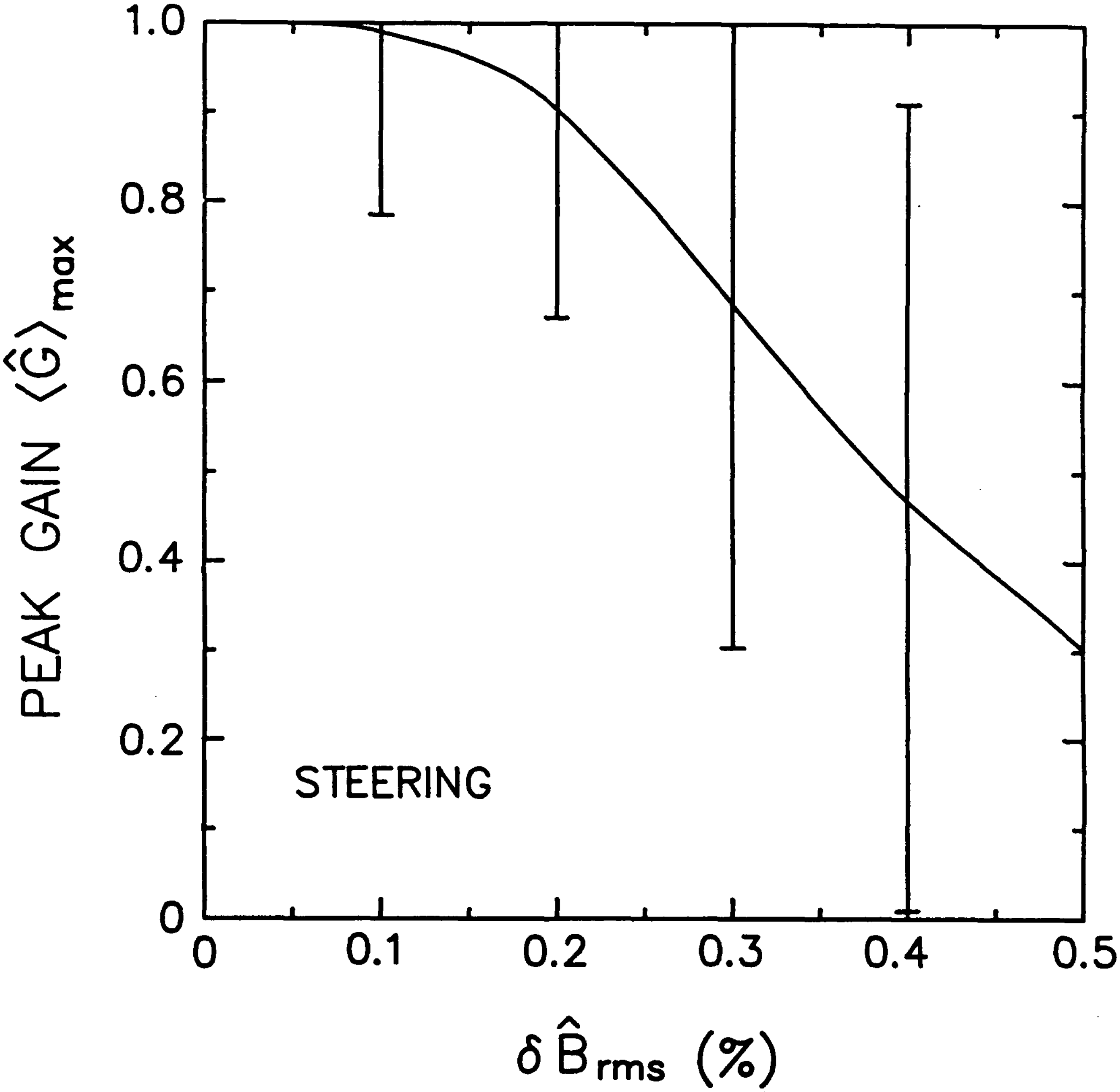

Fig. 7. Peak gain $\langle \hat{G} \rangle_{max}$ versus normalized rms field error $\delta \hat{B}_{rms}$ with steering for a linearly polarized wiggler with $B_w = 5.4$ kG, $\lambda_w = 2.8$ cm, $N = 130$ and $\gamma = 350$ in the limit $k_\beta = 0$. The bars denote one standard deviation as obtained from the gain variance.